

# Geodesic deviation, Raychaudhuri equation, and tidal forces in modified gravity with an arbitrary curvature-matter coupling

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The geodesic deviation equation, describing the relative accelerations of nearby particles, and the Raychaudhuri equation, giving the evolution of the kinematical quantities associated with deformations (expansion, shear and rotation) are considered in the framework of modified theories of gravity with an arbitrary curvature-matter coupling, by taking into account the effects of the extra force. As a physical application of the geodesic deviation equation the modifications of the tidal forces due to the supplementary curvature-matter coupling are obtained in the weak field approximation. The tidal motion of test particles is directly influenced not only by the gradient of the extra force, which is basically determined by the gradient of the Ricci scalar, but also by an explicit coupling between the velocity and the Riemann curvature tensor. As a specific example, the expression of the Roche limit (the orbital distance at which a satellite will begin to be tidally torn apart by the body it is orbiting) is also obtained for this class of models.

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## I. INTRODUCTION

The strong observational evidence that recently our universe has entered an accelerated expansion phase [1], and the astonishing result that around 95–96% of the content of the Universe is in the form of dark energy and dark matter, respectively, with only about 4–5% being represented by baryonic matter [2], has shown the limitations of standard general relativity. Hence, despite the remarkable success of general relativity at the solar system scale, on a galactic and cosmological scale gravitational theories face two fundamental problems: the dark matter problem, and the dark energy problem, respectively. Although in recent years many different approaches have been proposed to explain the observational results of cosmology, a satisfactory model has yet to be obtained.

However, a promising way to explain the observational data is to assume that at large scales the Einstein gravity model of general relativity breaks down, and a more general action describes the gravitational field. Theoretical models in which the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar  $R$ , first proposed in [3], have recently been extensively investigated. Cosmic acceleration can be explained by  $f(R)$  gravity [4], and viable cosmological models can be found [5]. For a review of  $f(R)$  generalized gravity models see [6]. The possibility that the galactic dynamic of mas-

sive test particles can be understood, without the need for dark matter, was also extensively considered in the framework of  $f(R)$  gravity [7]. In the context of the Solar System regime, local tests in the weak field approximation are also to be retained by assuming the chameleon mechanism [8], where the mass of the chameleon scalar field depends on the local background matter density, i.e., in regions of high matter density the chameleon scalar field is massive. However, it has recently been shown that in the hybrid metric-Palatini gravitational theory, which consists of the superposition of the metric Einstein-Hilbert Lagrangian with an  $f(R)$  term constructed à la Palatini [9], that even if the scalar field is very light, the theory passes the Solar System observational constraints. Therefore the model predicts the existence of a long-range scalar field, modifying the cosmological and galactic dynamics.

However, most of the generalizations of standard general relativity concentrated only on the geometric part of the action, and assumed that the matter part is unchanged, that is, in the total Lagrangian the matter term was considered as a simple additive term. This point of view severely limits the possibilities of the matter-geometry interaction, restricting the degrees of freedom of the gravitational theories. From a physical point of view, generalized gravitational models, involving curvature-matter interactions, cannot be ruled out *a priori* [10]. In this context, a generalization of  $f(R)$  gravity was introduced in [11], and extended in [12], by including in the theory an explicit coupling of an arbitrary function of the curvature scalar,  $R$ , with the matter Lagrangian density. As a result of the coupling, the motion of the massive particles is non-geodesic, and an extra force, or-

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thogonal to the four-velocity, arises. This class of models of modified gravity can be denoted as generalized gravity models with a linear curvature-matter coupling, and they have been extensively studied recently [13]. For a review of modified gravity models with curvature-matter coupling see [14].

Similar couplings between gravitation and matter have also been considered, as possible explanations for the accelerated expansion of the universe and of the dark energy, in [15]. An interesting extension of standard general relativity has also been proposed, namely, the  $f(R, T)$  modified theories of gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar  $R$  and of the trace of the stress-energy tensor  $T$  [16]. It is interesting to note that the dependence from  $T$  may be induced by exotic imperfect fluids or quantum effects (conformal anomaly). This extended theory of gravity may be considered as a relativistically covariant model of interacting dark energy. It was further argued that the new matter and time dependent terms in the gravitational field equations play the role of an effective cosmological constant.

The above models were further generalized in a recent proposal where the gravitational action is given by an arbitrary function of the Ricci scalar  $R$  and of the Lagrangian density of the matter  $L_m$  [17], and with a generalized scalar field and kinetic term dependences [18]. Thus,  $f(R, L_m)$  represents the natural generalization of the models with linear matter coupling, as well as the most general extension of the standard Hilbert action for the gravitational field,  $S = \int [R + L_m] \sqrt{-g} d^4x$ . In this class of models the energy-momentum tensor of the matter is generally not conserved, and the motion is non-geodesic. In the particular case in which the Lagrange function of the matter is a function of the energy density of the matter only, the equations of motion of test particles can be obtained, by using a variational principle.

It is the purpose of the present paper to investigate some other interesting properties of the motion of test particles in gravity models with an arbitrary curvature-matter coupling. The weak field limit of the model is carefully analyzed, and it is shown that in first order in both Ricci scalar and matter energy density one obtains the Newtonian Poisson equation in the presence of an effective cosmological constant. The equation of geodesic deviation and the Raychaudhuri equation are also formulated by explicitly including in the equations the effects of the curvature-matter coupling and of the extra force. Some of the physical implications of the geodesic deviation equation, namely, the problem of the tidal forces in this class of models is also considered, and the generalization of the Roche limit is obtained.

The present paper is organized as follows. The field equations and the equations of motion of the model are derived in Section II. The geodesic deviation equation (the Jacobi equation) of the test particles, as well as the Raychaudhuri equation of the model, are obtained in Section III. Some physical applications of the geodesic

deviation equation are discussed in Section IV. We discuss and conclude our results in Section V. In the present paper we consider a system of units with  $8\pi G = c = 1$ , and we follow the Landau-Lifshitz [19] conventions for the metric signature and the definition of the curvature tensor.

## II. FIELD AND MOTION EQUATIONS WITH AN ARBITRARY CURVATURE-MATTER COUPLING

The most general action for a  $f(R, L_m)$  type modified theory of gravity involving an arbitrary coupling between matter and curvature is given by [12]

$$S = \int f(R, L_m) \sqrt{-g} d^4x, \quad (1)$$

where  $f(R, L_m)$  is an arbitrary function of the Ricci scalar  $R$ , and of the Lagrangian density corresponding to matter,  $L_m$ . The only requirement for the function  $f(R, L_m)$  is to be an analytical function of  $R$  and  $L_m$ , respectively, that is, it must possess a Taylor series expansion about any point. The matter energy-momentum tensor  $T_{\mu\nu}$  is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \quad (2)$$

By assuming that the Lagrangian density  $L_m$  of the matter depends only on the metric tensor components, and not on its derivatives, we obtain  $T_{\mu\nu} = L_m g_{\mu\nu} - 2\partial L_m / \partial g^{\mu\nu}$ .

Varying the action with respect to the metric tensor  $g_{\mu\nu}$  we obtain the field equations of the model as

$$f_R(R, L_m) R_{\mu\nu} + \hat{P}_{\mu\nu} f_R(R, L_m) - \frac{1}{2} [f(R, L_m) - f_{L_m}(R, L_m) L_m] g_{\mu\nu} = \frac{1}{2} f_{L_m}(R, L_m) T_{\mu\nu}, \quad (3)$$

where we have denoted  $f_R(R, L_m) = \partial f(R, L_m) / \partial R$  and  $f_{L_m}(R, L_m) = \partial f(R, L_m) / \partial L_m$ , respectively, and we have introduced the operator  $\hat{P}_{\mu\nu}$ , defined as

$$\hat{P}_{\mu\nu} = g_{\mu\nu} \square - \nabla_\mu \nabla_\nu, \quad (4)$$

with  $\square = \nabla_\mu \nabla^\mu$ . The operator  $\hat{P}_{\mu\nu}$  has the property  $\hat{P}_\mu^\mu = 3\square$ .

By contracting the field equations Eq. (3), we obtain the scalar equation

$$3\square f_R + f_R R - 2f = \left( \frac{1}{2} T - 2L_m \right) f_{L_m}, \quad (5)$$

where  $T = T_\mu^\mu$  is the trace of the matter energy-momentum tensor. By eliminating the term  $\square f_R(R, L_m)$

between Eq. (3) and Eq. (5), we can reformulate the field equations as

$$R_{\mu\nu} = \Lambda(R, L_m) g_{\mu\nu} + \frac{1}{f_R(R, L_m)} \nabla_\mu \nabla_\nu f_R(R, L_m) + \Phi(R, L_m) \left( T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right), \quad (6)$$

where we have denoted

$$\Lambda(R, L_m) = \frac{2f_R(R, L_m)R - f(R, L_m) + f_{L_m}(R, L_m)L_m}{6f_R(R, L_m)}, \quad (7)$$

and

$$\Phi(R, L_m) = \frac{f_{L_m}(R, L_m)}{f_R(R, L_m)}, \quad (8)$$

respectively.

By taking the covariant divergence of Eq. (3), we obtain for the divergence of the energy-momentum tensor  $T_{\mu\nu}$  the following relationship

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu \ln[f_{L_m}(R, L_m)] (L_m g_{\mu\nu} - T_{\mu\nu}) \\ &= 2\nabla^\mu \ln[f_{L_m}(R, L_m)] \frac{\partial L_m}{\partial g^{\mu\nu}}. \end{aligned} \quad (9)$$

Now, assuming that the matter Lagrangian is a function of the rest mass density  $\rho$  of the matter only, from Eq. (9) we obtain explicitly the equation of motion of the test particles in the  $f(R, L_m)$  gravity model as

$$\frac{D^2 x^\mu}{ds^2} = U^\nu \nabla_\nu U^\mu = \frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\lambda}^\mu U^\nu U^\lambda = f^\mu, \quad (10)$$

where the wordline parameter  $s$  is taken as the proper time,  $U^\mu = dx^\mu/ds$  is the four-velocity of the particle,  $\Gamma_{\sigma\beta}^\nu$  are the Christoffel symbols associated to the metric, and the extra-force  $f^\mu$  is defined as

$$f^\mu = -\nabla_\nu \ln \left[ f_{L_m}(R, L_m) \frac{dL_m(\rho)}{d\rho} \right] (U^\mu U^\nu - g^{\mu\nu}). \quad (11)$$

The extra-force  $f^\mu$ , generated by the curvature-matter coupling, is perpendicular to the four-velocity,  $f^\mu u_\mu = 0$ . Due to the presence of the extra-force  $f^\mu$ , the motion of the test particles in modified theories of gravity with an arbitrary coupling between matter and curvature is non-geodesic. From the relation  $U_\mu \nabla_\nu U^\mu \equiv 0$  it follows that the force  $f^\mu$  is always perpendicular to the velocity, so that  $U_\mu f^\mu = 0$ .

The generalized equations of motion Eq. (10) can be derived from the Kahil-Bazanski Lagrangian [20],

$$L_p = g_{\mu\nu} U^\mu \frac{D\eta^\nu}{ds} + f_\mu \eta^\mu = g_{\mu\nu} U^\mu \dot{\eta}^\nu + f_\mu \eta^\mu, \quad (12)$$

where  $\eta^\nu$  is an arbitrary four-vector, which can be taken, for example, as the deviation vector (see Section IV), and we have denoted, for simplicity,

$$\frac{D\eta^\nu}{ds} = \frac{d\eta^\nu}{ds} + \Gamma_{\sigma\beta}^\nu \eta^\sigma U^\beta = \dot{\eta}^\nu. \quad (13)$$

Then we obtain immediately  $\partial L_p / \partial \dot{\eta}^\sigma = U_\sigma$ , and  $\partial L_p / \partial \eta^\sigma = \Gamma_{\sigma\beta}^\nu U_\nu U^\beta + f_\sigma$ , respectively. Finally, the Lagrange equations

$$\frac{d}{ds} \left( \frac{\partial L_p}{\partial \dot{\eta}^\nu} \right) - \frac{\partial L_p}{\partial \eta^\nu} = 0, \quad (14)$$

provide the equations of motion Eq. (10).

### III. WEAK FIELD LIMIT OF THE FIELD EQUATIONS IN $f(R, L_m)$ GRAVITY

Generally the matter Lagrangian  $L_m$  is a function of the matter energy density  $\rho$ , the pressure  $p$  as well as the other thermodynamic quantities, such as the specific entropy  $s$  or the baryon number  $n$ , so that  $L_m = L_m(\rho, p, s, n)$ . In the simple (but physically the most relevant) case in which the matter obeys a barotropic equation of state, so that the pressure is a function of the energy density of the matter only,  $p = p(\rho)$ , the matter Lagrangian becomes a function of the energy density only, and hence  $L_m = L_m(\rho)$ . Then, the matter Lagrangian is given by [12]

$$L_m(\rho) = \rho \left( 1 + \int_0^p \frac{dp}{\rho} \right) - p(\rho), \quad (15)$$

while the energy-momentum tensor can be written as

$$T^{\mu\nu} = [\rho + p(\rho) + \rho \Pi(\rho)] U^\mu U^\nu - p(\rho) g^{\mu\nu}, \quad (16)$$

respectively, where

$$\Pi(\rho) = \int_0^p \frac{p}{\rho^2} d\rho = \int_0^p \frac{dp}{\rho} - \frac{p(\rho)}{\rho}. \quad (17)$$

The expression  $\Pi(\rho) + p(\rho)/\rho$  represents the specific enthalpy of the fluid. From a physical point of view  $\Pi(\rho)$  can be interpreted as the elastic (deformation) potential energy of the body, and therefore Eq. (16) corresponds to the energy-momentum tensor of a compressible elastic isotropic system.

Next, we consider the weak field limit of the gravitational field equations. First, we assume that  $\rho \gg p$ , and therefore we systematically neglect the pressure term. Then, from Eq. (15) it follows that  $L_m = \rho$ , so that the energy-momentum tensor is given by  $T_{\mu\nu} = \rho U_\mu U_\nu$ .

Secondly, we consider non-relativistic macroscopic motion. Consequently we can neglect the spatial components in the four-velocity, and retain only the time component, so that  $U^\mu = U^\mu \approx (1, 0, 0, 0)$ . Since in the weak field limit one can omit all time derivatives as well as the terms containing the products of the Christoffel symbols, we obtain for  $R_{00}$  the expression [19]

$$R_{00} = R_{00}^0 = -\frac{\eta^{il} \partial^2 \phi}{\partial x^i \partial x^l} = \Delta \phi. \quad (18)$$

Similarly, we have  $\Gamma_{0,i0} = \Gamma_{i0}^0 = \partial \phi / \partial x^i$ ,  $i = 1, 2, 3$  and  $R_{i0}^i = \partial^2 \phi / (\partial x^i)^2$ ,  $i = 1, 2, 3$  (no summation upon the index  $i$ ). Therefore we obtain  $R \approx 2\Delta \phi$ .

Now, multiplying Eq. (6) with  $U^\mu U^\nu$  gives the scalar equation

$$U^\mu U^\nu R_{\mu\nu} = \frac{1}{f_R(R, L_m)} U^\mu U^\nu \nabla_\mu \nabla_\nu f_R(R, L_m) + \Lambda(R, L_m) + \Phi(R, L_m) \left( U^\mu U^\nu T_{\mu\nu} - \frac{1}{3} T \right). \quad (19)$$

In the weak field limit and for a static geometry, Eq. (19) immediately provides the following relationship

$$\Delta\phi = \Lambda(R, L_m) + \frac{2}{3} \Phi(R, L_m) \rho. \quad (20)$$

The left hand side of Eq. (20) is first order in  $1/c^2$ . Therefore, we will estimate the right hand side of Eq. (20) in the same order of approximation. By using a Taylor series expansion to first order of approximation we obtain first for the function  $\Lambda(R, L_m)$  the approximate representation

$$\Lambda(R, L_m) = \alpha + \beta R + \gamma L_m, \quad (21)$$

where

$$\alpha = -\frac{f(0, 0)}{6f_R(0, 0)}, \quad (22)$$

$$\beta = \frac{f_R^3(0, 0) + f(0, 0)f_{RR}(0, 0)f_R(0, 0)}{6f_R^3(0, 0)}, \quad (23)$$

and

$$\gamma = \frac{f_{RL_m}(0, 0)f(0, 0)}{6f_R^2(0, 0)}, \quad (24)$$

respectively.

For the function  $(2/3)\Phi(R, L_m)\rho$  we obtain

$$\frac{2}{3} \Phi(R, L_m) \rho \approx \frac{2}{3} \delta \rho, \quad (25)$$

where

$$\delta = \frac{f_{L_m}(0, 0)}{f_R(0, 0)}. \quad (26)$$

With these approximations Eq. (20) becomes

$$\Delta\phi \approx \alpha + \beta R + \left( \gamma + \frac{2}{3} \delta \right) \rho. \quad (27)$$

By substituting into Eq. (27) the weak field approximation of  $R$ ,  $R \approx 2\Delta\phi$ , gives the generalized Poisson equation in modified theories of gravity with an arbitrary matter-geometry coupling as

$$\Delta\phi \approx \frac{\gamma + 2\delta/3}{1 - 2\beta} \rho + \Lambda_0, \quad (28)$$

where the condition  $\beta \neq 1/2$  must hold for all  $\rho$ , and where we have denoted  $\Lambda_0 = \alpha/(1 - 2\beta)$ .

In order to obtain the correct limit of the Newtonian Poisson equation, the function  $f$  and its derivatives estimated at the point  $(0, 0)$  must satisfy the condition  $(\gamma + 2\delta/3)/(1 - 2\beta) = 1/2$ . The constant  $\Lambda_0$  plays the role of an effective cosmological constant, which is naturally generated in the present model. In most of the astrophysical applications  $\Lambda_0$  can be neglected. Therefore in the first approximation for the potential of the gravitational field of a single particle of mass  $m$  we obtain  $\phi(r) = -m/8\pi r$ , which is the expression of the standard Newtonian potential. As a result, in the equation of motion of the test particles one can take  $\vec{a} = -\nabla\phi = \vec{a}_N$ , where  $\vec{a}_N$  is the Newtonian acceleration of the particle.

#### IV. GEODESIC DEVIATION AND THE RAYCHAUDHURY EQUATION WITH AN ARBITRARY CURVATURE-MATTER COUPLING

As one can see from Eq. (10), the proper acceleration  $d^2x^\mu/ds^2$  is not a covariant object. In particular, its vanishing or non-vanishing has no observer-independent meaning. In contrast, the relative acceleration between worldlines is a covariant quantity, and its vanishing or non-vanishing, does not depend on the frame of reference.

Consider a one-parameter congruence of curves  $x^\mu(s; \lambda)$ , so that for each  $\lambda = \lambda_0 = \text{constant}$ ,  $x^\mu(s; \lambda_0)$  satisfies Eq. (10). We suppose the parametrization to be smooth, and hence we can introduce the tangent vector fields along the trajectories of the particles as  $U^\mu = \partial x^\mu(s; \lambda)/\partial s$  and  $n^\mu = \partial x^\mu(s; \lambda)/\partial \lambda$ , respectively. We also introduce the four-vector

$$\eta^\mu = \left[ \frac{\partial x^\mu(s; \lambda)}{\partial \lambda} \right] \delta\lambda \equiv n^\mu \delta\lambda, \quad (29)$$

joining points on infinitely close geodesics, corresponding to parameter values  $\lambda$  and  $\lambda + \delta\lambda$ , which have the same value of  $s$  [19, 21]. From the definition of  $U^\mu$  and  $n^\mu$  it follows that they satisfy the relation  $\partial U^\mu/\partial \lambda = \partial n^\mu/\partial s$ . Then it can be easily shown that  $n^\nu \nabla_\nu U^\mu = U^\nu \nabla_\nu n^\mu$  [19, 21]. Now consider the second derivative

$$\begin{aligned} \frac{D^2 n^\mu}{ds^2} &\equiv U^\nu \nabla_\nu (U^\alpha \nabla_\alpha n^\mu) = U^\nu \nabla_\nu (n^\alpha \nabla_\alpha U^\mu) \\ &= (\nabla_\nu \nabla_\alpha U^\mu) n^\alpha U^\nu + (\nabla_\nu n^\alpha) (\nabla_\alpha U^\mu) U^\nu. \end{aligned} \quad (30)$$

By changing the order of covariant differentiation by using the definition of the Riemann curvature tensor  $R^\mu_{\beta\nu\alpha}$ ,  $(\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) U^\mu = -R^\mu_{\beta\nu\alpha} U^\beta$  [19], Eq. (30) can be written as

$$\frac{D^2 n^\mu}{ds^2} = R^\mu_{\nu\alpha\beta} n^\alpha U^\beta U^\nu + \nabla_\alpha (U^\nu \nabla_\nu U^\mu) n^\alpha. \quad (31)$$

By taking into account Eq. (10), after multiplication with the constant factor  $\delta\lambda$ , we obtain the geodesic de-

viation equation (Jacobi equation) as [19, 21–23]

$$\frac{D^2 \eta^\mu}{ds^2} = R_{\nu\alpha\beta}^\mu \eta^\alpha U^\beta U^\nu + \eta^\alpha \nabla_\alpha f^\mu. \quad (32)$$

In the case  $f^\mu \equiv 0$  we reobtain the standard Jacobi equation, corresponding to the geodesic motion of test particles. The interest in the deviation vector  $\eta^\mu$  derives from the fact that if  $x_0^\mu(s) = x^\mu(s; \lambda_0)$  is a solution of Eq. (32), then to first order  $x_1^\mu(s) = x_0^\mu(s) + \eta^\mu$  is a solution as well, since  $x^\mu(s; \lambda_1) \approx x^\mu(s; \lambda_0) + n^\mu(s; \lambda_0) \delta\lambda \approx x^\mu(s; \lambda_0 + \delta\lambda)$ .

Note that the geodesic deviation equation Eq. (32) can also be derived from the Lagrangian [21]

$$L(\eta) = \frac{1}{2} g_{\mu\nu} \frac{D\eta^\mu}{ds} \frac{D\eta^\nu}{ds} + \frac{1}{2} R_{\mu\nu\alpha\beta} \eta^\mu \eta^\alpha U^\beta U^\nu + g_{\beta\mu} \eta^\beta \eta^\alpha \nabla_\alpha f^\mu. \quad (33)$$

In this Lagrangian the metric, connection and curvature are those of a given reference geodesic  $x_0^\mu(s)$ , with  $U^\mu(s) = \dot{x}_0^\mu(s)$  representing the four velocity along the

same geodesic. These quantities are the background variables. The  $\eta^\mu(s)$  are the independent Lagrangian generalized coordinates, which are to be varied in the action according to the Lagrange equations,

$$\frac{d}{ds} \frac{\partial L}{\partial (D\eta^\nu/ds)} - \frac{\partial L}{\partial \eta^\nu} = 0. \quad (34)$$

Then these Lagrange equations give again the equation of the geodesic deviation.

By taking into account the explicit form of the extra-force given by Eq. (11), in modified theories of gravity with a curvature-matter coupling, the geodesic deviation equation can be written as

$$\frac{D^2 \eta^\mu}{ds^2} = R_{\nu\alpha\beta}^\mu \eta^\alpha U^\beta U^\nu + \eta^\alpha \nabla_\alpha \left\{ \nabla_\nu \ln \left[ f_{L_m}(R, L_m) \frac{dL_m(\rho)}{d\rho} \right] (L_m g^{\mu\nu} - T^{\mu\nu}) \right\}.$$

Explicitly, the geodesic deviation equation becomes

$$\begin{aligned} \frac{D^2 \eta^\mu}{ds^2} = & R_{\nu\alpha\beta}^\mu \eta^\alpha U^\beta U^\nu + \eta^\alpha \left\{ \nabla_\alpha \nabla_\nu \ln \left[ f_{L_m}(R, L_m) \frac{dL_m(\rho)}{d\rho} \right] \right\} (L_m g^{\mu\nu} - T^{\mu\nu}) \\ & + \eta^\alpha \nabla_\nu \ln \left[ f_{L_m}(R, L_m) \frac{dL_m(\rho)}{d\rho} \right] (g^{\mu\nu} \nabla_\alpha L_m - \nabla_\alpha T^{\mu\nu}). \end{aligned} \quad (35)$$

As a specific example, consider the case of a linear coupling between curvature and matter [11], where the Lagrangian given by  $f(R, L_m) = f_1(R) + \lambda f_2(R) L_m$ , and  $f_i(R)$  (with  $i = 1, 2$ ) are arbitrary functions of the Ricci scalar  $R$  and the strength of the interaction between  $f_2(R)$  and the matter Lagrangian is characterized by a coupling constant. For this case, the geodesic deviation equation can be written as

$$\begin{aligned} \frac{D^2 \eta^\mu}{ds^2} = & R_{\nu\alpha\beta}^\mu \eta^\alpha U^\beta U^\nu + \lambda \frac{[1 + \lambda f_2(R)] d[\ln F_2(R)]/dR - 1}{[1 + \lambda f_2(R)]^2} (L_m g^{\mu\nu} - T^{\mu\nu}) \eta^\alpha \nabla_\alpha R \nabla_\nu R \\ & + \frac{\lambda F_2(R)}{1 + \lambda f_2(R)} (L_m g^{\mu\nu} - T^{\mu\nu}) \eta^\alpha \nabla_\alpha \nabla_\nu R + \frac{\lambda F_2(R)}{1 + \lambda f_2(R)} \eta^\alpha (g^{\mu\nu} \nabla_\alpha L_m - \nabla_\alpha T^{\mu\nu}) \nabla_\nu R. \end{aligned} \quad (36)$$

Note that a second order tensor  $\nabla_\nu U_\mu$  can be decomposed into symmetric and antisymmetric parts, and the symmetric part can be further decomposed into a trace and trace-free part. Thus, in general, we can write [22, 23]

$$\nabla_\mu U_\nu = \frac{1}{3} \theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} + \dot{U}_\mu U_\nu, \quad (37)$$

where  $h_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu$ ,  $\dot{U}_\mu = U^\nu \nabla_\nu U_\mu$ ,  $\theta = \nabla_\nu U^\nu$  is the expansion of the congruence of particles. The shear  $\sigma_{\mu\nu}$  is given by

$$\sigma_{\mu\nu} = \nabla_{(\mu} U_{\nu)} - \frac{1}{3} \theta h_{\mu\nu} - \dot{U}_{(\mu} U_{\nu)}, \quad (38)$$

where

$$\nabla_{(\mu} U_{\nu)} = \frac{1}{2} (\nabla_\nu U_\mu + \nabla_\mu U_\nu), \quad (39)$$

and the vorticity  $\omega_{\mu\nu}$  is defined as

$$\omega_{\mu\nu} = \nabla_{[\mu} U_{\nu]} - \dot{U}_{[\mu} U_{\nu]}, \quad (40)$$

respectively, where

$$\nabla_{[\mu} U_{\nu]} = \frac{1}{2} (\nabla_\nu U_\mu - \nabla_\mu U_\nu). \quad (41)$$

The term  $\dot{U}_\mu U_\nu$  takes into account the possible presence of other forces, which are orthogonal to the four-velocity, with four-acceleration given by  $\dot{U}_\mu = U^\nu \nabla_\nu U_\mu$ .



From the definition of the Riemann curvature we have

$$(\nabla_\nu \nabla_\alpha - \nabla_\alpha \nabla_\nu) U^\mu = -R^\mu_{\beta\nu\alpha} U^\beta. \quad (42)$$

By contracting with  $\mu = \nu$  and after multiplication with  $U^\alpha$  we obtain [23]

$$U^\alpha \nabla_\nu \nabla_\alpha U^\nu - U^\alpha \nabla_\alpha \theta = -R_{\alpha\beta} U^\alpha U^\beta. \quad (43)$$

The first term in this equation can be written as

$$U^\alpha \nabla_\nu \nabla_\alpha U^\nu = \nabla_\nu (U^\alpha \nabla_\alpha U^\nu) - (\nabla_\nu U_\alpha) (\nabla^\alpha U^\nu). \quad (44)$$

Hence we obtain the Raychaudhuri equation in the presence of an extra force as [22, 23]

$$\dot{\theta} + \frac{1}{3}\theta^2 + (\sigma^2 - \omega^2) = \nabla_\mu f^\mu + R_{\mu\nu} U^\mu U^\nu, \quad (45)$$

where  $\sigma^2 = \sigma_{\mu\nu} \sigma^{\mu\nu}$  and  $\omega^2 = \omega_{\mu\nu} \omega^{\mu\nu}$ , respectively.

With the use of the field equation Eq. (6), and the expression of the extra force, in modified theories of gravity theories with an arbitrary coupling between curvature and matter, the Raychaudhuri equation assumes the following generalised form

$$\begin{aligned} \dot{\theta} = & -\frac{1}{3}\theta^2 - (\sigma^2 - \omega^2) + \Lambda(R, L_m) \\ & + \nabla_\mu \left\{ \nabla_\nu \ln \left[ f_{L_m}(R, L_m) \frac{dL_m(\rho)}{d\rho} \right] (L_m g^{\mu\nu} - T^{\mu\nu}) \right\} \\ & + \frac{1}{f_R(R, L_m)} U^\mu U^\nu \nabla_\mu \nabla_\nu f_R(R, L_m) \\ & + \Phi(R, L_m) \left( T_{\mu\nu} U^\mu U^\nu - \frac{1}{3}T \right). \end{aligned} \quad (46)$$

The vorticity  $\omega_{\mu\nu}$  satisfies the equation [22, 23]

$$\dot{\omega}_{\mu\nu} = -\frac{2}{3}\theta\omega_{\mu\nu} - 2\sigma^\lambda_{[\mu}\omega_{\nu]\lambda} + \nabla_{[\mu}f_{\nu]}, \quad (47)$$

while the dynamics of the shear  $\sigma_{\mu\nu}$  is described by the equation [22, 23]

$$\begin{aligned} \dot{\sigma}_{\mu\nu} = & \frac{1}{2}h^\sigma_\mu h^\lambda_\nu R_{\sigma\lambda} + h^\lambda_\mu h^\sigma_\nu \nabla_{(\lambda} f_{\sigma)} - \frac{2}{3}\theta\sigma_{\mu\nu} - \omega_{\mu\lambda}\omega^\lambda_\nu \\ & - \sigma_{\mu\lambda}\sigma^\lambda_\nu - \frac{1}{3}h_{\mu\nu} \left( \omega^2 - \sigma^2 + \frac{1}{2}h^{\sigma\lambda}R_{\sigma\lambda} + \nabla_\mu f^\mu \right) \\ & - C_{\mu\sigma\nu\lambda} U^\sigma U^\lambda, \end{aligned} \quad (48)$$

where

$$C_{\mu\sigma\nu\lambda} = R_{\mu\sigma\nu\lambda} + g_{\mu[\lambda}R_{\nu]\sigma} + g_{\sigma[\nu}R_{\lambda]\mu} + \frac{Rg_{\mu[\nu}g_{\lambda]\sigma}}{3}, \quad (49)$$

is the Weyl tensor. With the use of the field equations Eq. (6) and of the expression of the extra force  $f^\mu$ , the evolution equations for the shear and vorticity can also be obtained explicitly for modified gravity with an arbitrary curvature-matter coupling.

## V. TIDAL FORCES WITH AN ARBITRARY CURVATURE-MATTER COUPLING

Tides are the manifestation of a gradient of the gravitational force field induced by a mass above an extended body or a system of particles. In the Solar System tidal perturbations act on compact bodies such as planets, moons and comets. On larger scales than the solar system, as in a galactic or cosmological context, one can observe tidal deformations or disruptions of a stellar cluster by a galaxy, or in galaxy encounters [24]. In the relativistic theories of gravitation, as well as in Newtonian gravity, a local system of coordinates can be chosen, which is inertial except for the presence of the tidal forces. In strong gravitational fields, relativistic tidal effects can lead to interesting phenomena, such as the emission of tidal gravitational waves [24]. Relativistic corrections to the Newtonian tidal accelerations caused by a massive rotating source, such as, for example, the Earth, could be determined experimentally, at least in principle, thus leading to the possibility of testing relativistic theories of gravitation by measuring such effects in a laboratory.

### A. There are no geodesic reference frames in $f(R, L_m)$ gravity

If the Christoffel symbols  $\Gamma^\alpha_{\mu\nu}$  associated to a given metric are symmetric, it is always possible to chose a coordinate system in which all the  $\Gamma^\alpha_{\mu\nu}$  become zero at a previously assigned point [19]. The corresponding coordinates are called Riemann normal coordinates, and the system of reference can be called a locally Minkowski system. Let the given point be the origin of the coordinate system, and let the initial values of the Christoffel symbols be equal to  $(\Gamma^\mu_{\alpha\beta})_0$ . In the neighborhood of the origin we introduce the coordinate transformation

$$x'^\mu = x^\mu + \frac{1}{2} \left( \Gamma^\mu_{\alpha\beta} \right)_0 x^\alpha x^\beta, \quad (50)$$

which provides

$$\left( \frac{\partial^2 x'^\sigma}{\partial x^\alpha \partial x^\beta} \right) \left( \frac{\partial x'^\mu}{\partial x'^\sigma} \right) = \left( \Gamma^\mu_{\alpha\beta} \right)_0. \quad (51)$$

Then, from the transformation law of the Christoffel symbols,

$$\begin{aligned} \left( \Gamma^\mu_{\alpha\beta} \right)_0 = & \Gamma'^{\mu}_{\lambda\sigma} \left( \frac{\partial x'^\nu}{\partial x'^\lambda} \right) \left( \frac{\partial x'^\lambda}{\partial x^\alpha} \right) \left( \frac{\partial x'^\sigma}{\partial x^\beta} \right) \\ & + \left( \frac{\partial^2 x'^\sigma}{\partial x^\alpha \partial x^\beta} \right) \left( \frac{\partial x'^\mu}{\partial x'^\sigma} \right) \end{aligned} \quad (52)$$

it follows that all the transformed Christoffel symbols  $\Gamma'^{\mu}_{\lambda\sigma}$  are zero.

One can also show that by a suitable choice of the coordinate system one can make all the  $\Gamma^\mu_{\alpha\beta}$  go to zero not only in a point, but all along a given world line [19]. At

the same time with the vanishing of the Christoffel symbols, the metric  $g_{\mu\nu}$  can be reduced, at a given point, to its diagonal (Minkowskian) form,  $g_{\mu\nu} = \eta_{\mu\nu}$ , and consequently in the given point the covariant derivatives coincide with the ordinary partial derivatives,  $\nabla_\mu = \partial/\partial x^\mu$  [19, 22, 24, 25].

In such a coordinate system the equation of motion Eq. (10) takes the form

$$\left. \frac{d^2 x^\mu}{dt^2} \right|_{\text{geod}} = f^\mu|_{\text{geod}}, \quad (53)$$

where for small particle velocities  $ds \approx dt$ , where  $t$  is the time coordinate.

Therefore, in theories of gravity with an arbitrary curvature-matter coupling, despite the fact that one can locally cancel the Christoffel symbols, and can introduce a flat (Minkowskian) metric, due to the presence of the extra-force  $f^\mu$ , which generally cannot be reduced to zero, the motion of the test particles is non-inertial, and they will always experience a supplementary acceleration induced by the presence of the coupling between matter and curvature.

### B. Tidal forces in $f(R, L_m)$ gravity

In the following we will denote a reference frame in which all the Christoffel symbols vanish by a prime. In such a system one can always take  $\eta'^0 = 0$ , which means that the particle accelerations are compared at equal

times.  $\eta'^i$  is then the displacement of the particle from the origin [25]. Moreover, in the static/stationary case, in which the metric, the Ricci scalar and the thermodynamic parameters of the matter do not depend on time,  $f'^0 = 0$ . With the use of the equation of motion this condition implies  $U'^0 = \text{constant} = 1$ . Therefore, with these assumptions, the equation of the geodesic deviation (the Jacobi equation) takes the form

$$\frac{d^2 \eta'^i}{dt'^2} = R'^i_{0l0} \eta'^l + R'^i_{jlm} \eta'^l U'^j U'^m + \eta'^l \frac{\partial f'^i}{\partial x'^l}. \quad (54)$$

Equation (54) can be reformulated as

$$F'^i = \frac{d^2 \eta'^i}{dt'^2} = K'^i_j \eta'^j, \quad (55)$$

where  $F'^i$  is the tidal force, and we have introduced the generalized tidal matrix  $K'^i_l$  [24], which is defined as

$$K'^i_j = R'^i_{0j0} + R'^i_{kjm} U'^k U'^m + \frac{\partial f'^i}{\partial x'^j}. \quad (56)$$

The tidal force has the property  $\partial F'^i / \partial \eta'^j = K'^i_j$ , and its divergence is given by  $\partial F'^i / \partial \eta'^i = K$ , where the trace  $K$  of the tidal matrix is

$$K = K^i_i = R'^i_{00} + R'^i_{km} U'^k U'^m + \frac{\partial f'^j}{\partial x'^j}. \quad (57)$$

With the use of the gravitational field equations Eq. (6) we can express  $K$  as

$$\begin{aligned} K = & \Lambda(R', L'_m) \eta_{00} + \frac{1}{f_R(R', L'_m)} \frac{\partial^2}{\partial t'^2} f_R(R', L'_m) + \Phi(R', L'_m) \left( T'_{00} - \frac{1}{3} T' \eta_{00} \right) + \Lambda(R', L'_m) \eta_{km} U'^k U'^m \\ & + \frac{1}{f_R(R', L'_m)} U'^k U'^m \frac{\partial^2}{\partial x'^k \partial x'^m} f_R(R', L'_m) + \phi(R', L'_m) \left( T'_{km} - \frac{1}{3} T' \eta_{km} \right) U'^k U'^m \\ & + \frac{\partial}{\partial x'^k} \left\{ \frac{\partial}{\partial x'^m} \ln \left[ f_{L'_m}(R', L'_m) \frac{dL'_m(\rho)}{d\rho} \right] (L'_m \eta'^{km} - T'^{km}) \right\}. \end{aligned} \quad (58)$$

Then, since in the Newtonian limit one can omit all time derivatives, we obtain for  $R'^i_{0l0}$  the expression  $R'^i_{0l0} = \partial^2 \phi / \partial x^i \partial x^l$  [25], where, for simplicity, in the following we will omit the primes for the geometrical and physical quantities in the Newtonian approximation. In Newtonian gravity  $\tau_{il} = -\partial^2 \phi / \partial x^i \partial x^l$  represents the Newtonian tidal tensor [25]. Therefore, in modified gravity with a curvature-matter coupling we obtain the tidal acceleration of the test particles as

$$\frac{d^2 \eta^i}{dt^2} = F^i \approx \frac{\partial^2 \phi}{\partial x^i \partial x^l} \eta^l + R^i_{jlm} \eta^l V^j V^m + \eta^l \frac{\partial f^i}{\partial x^l}, \quad (59)$$

where  $V^j$  and  $V^m$  are the Newtonian three-dimensional

velocities. In the Newtonian approximation, in modified theories of gravity with a curvature-matter coupling the tidal force tensor is defined as

$$\frac{\partial F^i}{\partial \eta^l} = \frac{\partial^2 \phi}{\partial x^i \partial x^l} + R^i_{jlm} V^j V^m + \frac{\partial f^i}{\partial x^l}, \quad (60)$$

and its trace gives the generalized Poisson equation,

$$\frac{\partial F^i}{\partial \eta^i} = \Delta \phi + R_{jm} V^j V^m + \frac{\partial f^i}{\partial x^i}. \quad (61)$$

### C. The Roche limit in modified gravity with an arbitrary curvature-matter coupling

In Newtonian gravity, the spherical potential of a given particle with mass  $M$  is  $\phi(r) = -M/8\pi r$ . By choosing a frame of reference so that the  $x$ -axis passes through the particle's position, corresponding to the radial spherical coordinate, that is,  $(x = r, y = 0, z = 0)$ , the Newtonian tidal tensor is diagonal, and has the only non-zero components

$$\tau_{ii} = \text{diag} \left( \frac{2M}{8\pi r^3}, -\frac{M}{8\pi r^3}, -\frac{M}{8\pi r^3} \right). \quad (62)$$

The Newtonian tidal force  $\vec{F}_t$  can be written as  $F_{tx} = 2M\Delta x/8\pi r^3$ ,  $F_{ty} = -GM\Delta y/8\pi r^3$  and  $F_{tz} = -M\Delta z/8\pi r^3$ , respectively [25]. These results can be used to derive the generalization of the Roche limit in modified gravity with an arbitrary coupling between matter and curvature.

The Roche limit is the closest distance  $r_{\text{Roche}}$  that a celestial object with mass  $m$ , radius  $R_m$  and density  $\rho_m$ , held together only by its own gravity, can come to a massive body of mass  $M$ , radius  $R_M$  and density  $\rho_M$ , respectively, without being pulled apart by the massive object's tidal (gravitational) force [24]. For simplicity we will consider  $M \gg m$ , so that the center of mass of the system nearly coincides with the geometrical center of the mass  $M$ .

The elementary Newtonian theory of this process is as follows. Consider a small mass  $\Delta m$  located at the surface of the small object of mass  $m$ . There are two forces acting on  $\Delta m$ , the gravitational attraction of the mass  $m$ , given by

$$F_G = \frac{m\Delta m}{8\pi R_m^2}, \quad (63)$$

and the tidal force exerted by the massive object, which is given by

$$F_t = \frac{M\Delta m R}{8\pi r^3}, \quad (64)$$

where  $r$  is the distance between the centers of the two celestial bodies. The Roche limit is reached at the distance  $r = r_{\text{Roche}}$ , when the gravitational force and the tidal force exactly balance each other,  $F_G = F_t$ , thus giving [24]

$$r_{\text{Roche}} = R_m \left( \frac{M}{m} \right)^{1/3} = 2^{1/3} R_M \left( \frac{\rho_M}{\rho_m} \right)^{1/3}. \quad (65)$$

In modified gravity with a curvature-matter coupling the equilibrium between gravitational and tidal forces occurs at a distance  $r_{\text{Roche}}$  given by the equation

$$\left( \frac{M}{8\pi r_{\text{Roche}}^3} + R_{jrm}^r V^j V^m + \frac{\partial f^r}{\partial r} \right) R_m = \frac{m}{8\pi R_m^2} + f^r, \quad (66)$$

where  $f^r$  is the radial component of the extra-force, which modifies the Newtonian gravitational force, and the curvature tensor  $R_{jrm}^r$  (no summation upon  $r$ ) must be evaluated in the coordinate system in which the Newtonian tidal tensor is diagonal. Hence we obtain the generalized Roche limit in the presence of arbitrary geometry-matter coupling as

$$r_{\text{Roche}} \approx R_m \left( \frac{M}{m} \right)^{1/3} \times \left[ 1 + \frac{8\pi R_m^3}{3m} \left( R_{jrm}^r V^j V^m + \frac{\partial f^r}{\partial r} \right) - \frac{8\pi R_m^2}{3m} f^r \right], \quad (67)$$

where we have assumed that the gravitational effects due to the coupling between matter and curvature are small as compared to the Newtonian ones.

## VI. CONCLUSIONS

In the present paper we have developed some of the basic theoretical tools necessary to investigate the properties of gravitational models with direct curvature-matter interaction. In particular, we have obtained the Raychaudhuri equation, which can be used to analyze the existence of black holes in these types of theories, and to rigorously formulate the singularity theorems. The presence of the extra-force in the gravitational model leads to the appearance of some new terms in the Raychaudhuri and the geodesic deviation equation, terms which are a direct consequence of the curvature-matter coupling. Therefore the presence of the extra force can have a significant effect on the formation of massive astrophysical objects by gravitational collapse. However, we have to mention in the present gravity model the full non-linear theory contains higher than two spacetime derivatives in the field equations. Hence modified gravity theories with geometry-matter coupling can be considered as effective theories only apart from the special Einstein case.

The geodesic equation can be used to study the effects of the generalized tidal forces, which could lead to the possibility of observationally testing the model through the observational effects of the tides due to an extended mass distribution. Typical situations in which the effects of the tides are of major importance are galactic encounters, globular clusters under the influence of the galactic mass distribution, and Oort cloud perturbations by the galactic field [26]. It was suggested that the compressive tidal field in the center of the flat core of early type galaxies and of the ultra-luminous galaxies compresses the molecular clouds, producing the dense gas observed in the center of these galaxies. The curvature-matter interaction modifies the nature of the tidal forces. Therefore the observational study of the tidal forces may give some insights in the fundamental aspects of the gravitational interaction.



On the other hand the Newtonian limit of the geodesic deviation equation Eq. (59) shows that the tidal motion of test particles is directly influenced not only by the gradient of the extra force  $f$ , which is basically determined by the gradient of the Ricci scalar, but also by an explicit coupling between the velocity and the Riemann curvature tensor. The possibility that the Pioneer anomaly [27] is due to the presence of the extra force due to the geometry-matter coupling was analyzed in [11]. However, Eq. (59) shows that this effect may also be velocity dependent, leading to the possibility of the detection of these type of extra gravitational effects by the method of the Doppler tracking of a spacecraft. On the other hand, recent researches on the Pioneer anomaly [28] have given

convincing arguments against its modified gravity origin.

To conclude, the present study opens further possibilities for the experimental and observational investigation of the possible coupling between curvature and matter, and for a large class of modified gravity theories.

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